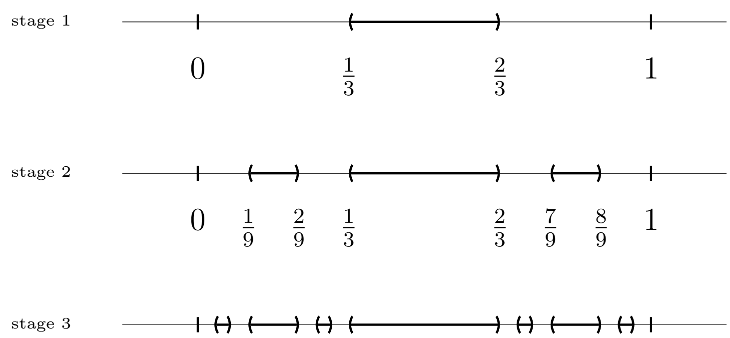
Cantor-Lebesgue function (uniformly continuous, not Lipschitz)

Construction of Cantor set. Let $I = [0,1]$. The first stage of the construction is to subdivide $[0,1]$ into thirds and remove the interior of the middle third; that is remove the open interval $(\frac{1}{3}, \frac{2}{3})$. Each successive step of the construction is essentially the same. Thus, at the second stage, we subdivide each of the remaining two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into thirds and remove the interiors, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, of the middle thirds. We continue the construction for each of the remaining intervals. The sets removed in the first three successive stages are indicated below by darkened intervals:



For each $k = 1, 2, \ldots$, let C_k denote the union of the interval left at the kth stage. The set $C = \bigcap_{k=1}^{\infty} C_k$ is called the **Cantor set.** Note that each C_k consists of 2^k closed interval, each of length 3^{-k}, and that C contains the endpoints of all these intervals. Any point of C belongs to an interval in C_k for every k and is therefore a limit point of the endpoints of the intervals, e.g. $\frac{2}{3} = \frac{2\,3^{k-1}}{3^k} = \lim_{k\to\infty} \frac{2\,3^{k-1}+1}{3^k}$ and note that $\frac{2\,3^{k-1}+1}{3^k}$ is an endpoint of an interval in C_k. Let $D_k = I \setminus C_k$. Then D_k consists of the $2^k - 1$ open intervals I_j^k (ordered from left to right) removed in the first k stages of construction of the Cantor set. Let f_k be the continuous function on $[0,1]$ defined by $f_k(x) =$

$$\begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ j2^{-k} & \text{if } x \in I_j^k,\ j = 1, \ldots, 2^k - 1, \\ \text{linear on each interval of } C_k \end{cases}$$

Note that

(1) each f_k is monotone increasing,
(2) $\{I_j^k\}_{j=1}^{2^k-1} \subseteq \{I_m^l\}_{m=1}^{2^l-1}$ whenever $k \le l$, we have $f_{k+1} = f_k$ on I_j^k, $j = 1, \ldots, 2^k - 1$,
(3) $|f_k - f_{k+1}| < 2^{-k}$ for all k.

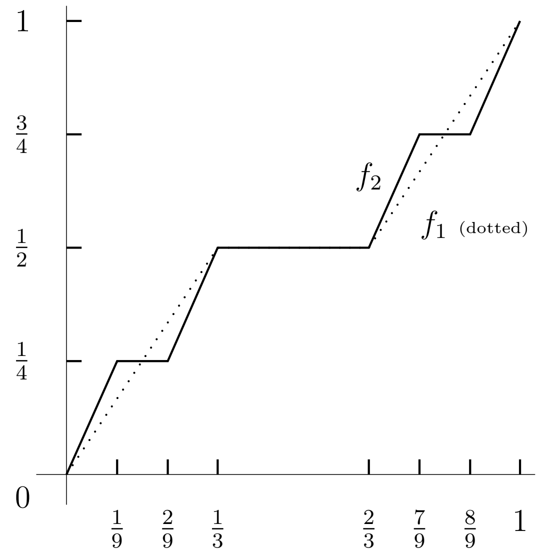
Hence, $\sum(f_k - f_{k+1})$ converges uniformly on $[0,1]$, and therefore, $\{f_k\}$ converges uniformly on $[0,1]$. Let $f = \lim_{k\to\infty} f_k$. Then $f(0) = 0$, $f(1) = 1$, f is monotone increasing and **(uniformly) continuous** on $[0,1]$, and f is constant on every interval removed in constructing C. (Alternative argument: By (3) and the Azelà-Ascoli Theorem, the sequence $\{f_k\}$ is a Cauchy sequence and it converges uniformly to a continuous function f.) This f is called the **Cantor-Lebesgue** function.

Note that
(1) $\frac{2}{3} = \frac{2\,3^{k-1}}{3^k}$, $\frac{2\,3^{k-1}+1}{3^k} \in C_k$ for all k,
(2) $f(\frac{2\,3^{k-1}}{3^k}) = f(\frac{2}{3}) = \frac{1}{2}$, and $f(\frac{2\,3^{k-1}+1}{3^k}) = \frac{1}{2} + \frac{1}{2^k}$.

Thus, we have

$$\frac{f(\frac{2\,3^{k-1}+1}{3^k}) - f(\frac{2}{3})}{\frac{2\,3^{k-1}+1}{3^k} - \frac{2}{3}} = \frac{\frac{1}{2} + \frac{1}{2^k} - \frac{1}{2}}{\frac{2\,3^{k-1}+1}{3^k} - \frac{2\,3^{k-1}}{3^k}} = \frac{3^k}{2^k} \to \infty \text{ as } k \text{ goes to } \infty$$

which implies that f is **Not Lipschitz** on $[0,1]$.


1
$\frac{3}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
0
$\frac{1}{9}$
$\frac{2}{9}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{7}{9}$
$\frac{8}{9}$
1
f_2
f_1 (dotted)